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COMBINATORIAL INTERACTION OF DISTURBANCES IN A SUPERSONIC BOUNDARY LAYER

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A nonlinear model of interaction of disturbances in the regime of coupled combinatorial relations is used to explain the dynamics of unstable waves arising due to introduction of controlled high-intensity disturbances into a supersonic boundary layer. The model includes effects of self-action and combinatorial interaction of the waves. The second-order model considered offers a rather accurate description of the streamwise dynamics of plane waves.

Introduction. The situation arising due to introduction of controlled disturbances of a rather high intensity into a boundary layer on a flat plate at a Mach number $\mathrm{M} = 2$ [1] is theoretically studied in the present paper. Kosinov et al. [1] called the downstream evolution of such disturbances "anomalous," since it differs significantly from the evolution of small perturbations observed earlier. It was established that quasi-two-dimensional disturbances are most unstable. The initial spectrum contains two wave packets with multiple frequencies (subharmonic frequency $f_1 = 10$ kHz and fundamental frequency $f_2 = 20$ kHz), the packet with the frequency f_1 dominates, and the greatest contribution to disturbance intensity is made by the plane wave. The two-dimensional character of the wave spectra remains unchanged further downstream, which is unusual because the growth rates of three-dimensional waves are significantly greater than the growth rates of plane waves in accordance with the linear theory. Significant (almost tenfold) amplification of disturbances with the frequency f_1 is observed; the growth rate of disturbances with the fundamental frequency f_2 is somewhat smaller but still much greater than linear. This is the reason for a 20% decrease in the distance between the leading edge and the point of laminar–turbulent transition. A significant steady distortion of the mean velocity profile across the boundary layer is registered: the profile becomes more filled in the near-wall region and less filled in the external region, which increases the boundary-layer thickness. It was assumed that the experimental results described are caused by the influence of a steady vortex generated by the source of controlled disturbances. It was also found that the phase velocities of disturbances are 30–40% greater than linear. Kosinov et al. [1] concluded that the features observed are caused by the nonlinear character of evolution of disturbances. The reasons for these anomalies were not established.

The objective of the present work is to explain the above-described features by nonlinear interaction of own travelling Tollmien–Schlichting waves without considering steady vortices, which have not been described theoretically. The analysis is performed within the framework of the weakly linear stability theory, which is used to explain the dynamics of disturbances at early stages of nonlinearity. This theory includes two already tested models: the model of interaction in resonance triads and a higher-order model of coupled combinatorial interactions.

It was shown in the experiments [2, 3] and calculations [4] that subharmonic instability is formed in a supersonic boundary layer with a moderate level of controlled disturbances. This instability can be described within the framework of resonance interactions of wave triplets. In this case, three-dimensional modes prevail in the disturbance spectrum, and filling of the spectrum is a cascade process of identification of three-dimensional subharmonics in a parametric domain. The effectiveness of resonance interactions in a subsonic boundary layer is known to decrease with increasing disturbance intensity or amplitude. This feature is also inherent in a supersonic boundary layer under the test conditions of [1]; however, a detailed discussion of the resonance model is not the subject of the present paper.

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In the present work, we consider the nonlinear evolution of high-intensity disturbances within the framework of the model of coupled combinatorial interactions of plane two-dimensional waves. The condition of the combinatorial type of interaction is a rather high value of the wave amplitude. In this case, both the self-action of the wave and the combinatorial interaction of two or more waves are possible. This nonlinear mechanism may be either alternative or complementary to the resonance mechanism in the course of energy redistribution in the amplitude–frequency spectrum of the excited flow. It is also important to study these interactions because they can occur in the process of identification of the deterministic frequency in evolution of wave packets of different nature, for instance, travelling and steady vortex disturbances, which can affect the whole process of excitation of waves whose frequencies may be other than multiple and may not satisfy the conditions of phase synchronism.

The special features of operation of the combinatorial mechanism of interaction of two plane waves are studied in the present paper within the framework of the weakly nonlinear stability theory. It should be noted that simulation of nonlinearity in subsonic boundary layer also started from studying the effects of self-action of finite-intensity waves [5, 6].

We denote the scale of the disturbance field by ε. Then, the mechanism considered can be described as follows. Self-action of the wave of order ε^2 leads to generation of zero secondary harmonics, which distort the mean flow field, and to induction of overtones with a doubled phase. Summary or difference secondary harmonics can be formed by combinatorial interaction of two waves. In the third order in terms of ε, interaction of secondary waves with initial disturbances determines the nonlinear evolution of the amplitudes of primary disturbances.

The evolution described is studied by means of integrating amplitude equations that are based on the known Landau equation [7, 8]. Thus, the mechanism of coupled interaction $O(\varepsilon^3)$ has a higher order of nonlinearity as compared with the resonance mechanism $O(\varepsilon^2)$. We consider all components of this interaction.

Basic Formulas and Methods of the Solution. The initial postulates of the nonlinear model for compressible boundary layers are described in detail in [4]. Following Gaponov and Maslennikova [4], we consider the disturbed fields of velocities u and v, density ρ_0, pressure p_0, and temperature T_0 of a compressible gas

$$u = U(Y) + \varepsilon u', \qquad v = \varepsilon v', \qquad \rho_0 = \rho(Y) + \varepsilon \varrho',$$
$$p_0 = P + \varepsilon p', \qquad T_0 = T(Y) + \varepsilon \Theta', \qquad p'/P = \varrho'/\rho + \Theta'/T \tag{1}$$

in a dimensionless coordinate system $X = x/\delta$, $Y = y/\delta$, where $\delta = \sqrt{\mu_{\rm e} x/(U_{\rm e}\rho_{\rm e})}$ (δ is the boundary-layer thickness and μ is the dynamic viscosity); the subscript "e" refers to parameters at the external boundary $y \geqslant \delta$; the primed and non-primed quantities are the fluctuating and mean components, respectively, of the corresponding quantities; the scale parameter is $\varepsilon \ll 1$. Normalization is performed by flow parameters at the external boundary. We introduce the Reynolds number based on these parameters: $\mathrm{Re} = \sqrt{x\rho_{\rm e}U_{\rm e}/\mu_{\rm e}}$. It should be noted that the dimensionless streamwise coordinate X coincides with the value of Re.

The steady undisturbed profiles of U, ρ, and T are found using the technique of Gaponov and Maslov [9] for $T = 1/\rho$.

The solution is constructed by the method of expansion in the small parameter ε and two-scale expansion of the x coordinate. In addition to the "fast" scale X, we introduce the "slow" scale $\xi = \varepsilon X$ characterizing the difference in variation of the disturbance phase and amplitude. The necessity of introducing the "slow" scale is caused by the large difference in the velocities mentioned ($\partial/\partial x = \partial/\partial X + \varepsilon\partial/\partial\xi$). We seek the solution for waves of the following form:

$$u'_j = A_j(\xi) u_j(Y) \exp(i\theta_j) + A_{-j}(\xi) u_{-j}(Y) \exp(-i\theta_j), \qquad j = 1, 2. \tag{2}$$

Here u'_j is the longitudinal component of velocity, A_j is the amplitude, which varies slowly along the streamwise coordinate, $u(Y)$ is the amplitude eigenfunction, the second term is a complex-conjugate quantity, and $\theta = \alpha X - \omega t$, where $\alpha = \alpha^r + i\alpha^i$ (α^r is the wavenumber and α^i is the growth rate) and $\omega = 2\pi f$. The subscripts $j = 1$ and 2 correspond to disturbances with the subharmonic frequency f_1 and fundamental frequency f_2, respectively.

We introduce the initial variables, in terms of which we seek the solution, in the form of the vector

$$\boldsymbol{z}^k = |u, u_Y, v, p, \Theta, \Theta_Y|, \qquad u_Y = \frac{du}{dY}, \qquad \Theta_Y = \frac{d\Theta}{dY}, \qquad k = 1, 2, \ldots, 6.$$

Substituting (1) and (2) into the full system of equations of motion and conservation for a compressible gas [9], we obtain the initial system for the disturbances within the framework of the weakly nonlinear theory (the subscript j is omitted):

$$L^k(\exp(i\theta)z^k) = F^k. \tag{3}$$

Here

$$L^1(\exp(i\theta)z^1) \equiv [z^2 - z^1_Y]\exp(i\theta) = 0,$$

$$L^2(\exp(i\theta)z^2) \equiv [\rho((-i\omega + i\alpha U)z^1 + U_Y z^3) + i\alpha z^4 - (\mu/\mathrm{Re})z^2_Y]\exp(i\theta) = 0,$$

$$L^3(\exp(i\theta)z^3) \equiv [\rho((-i\omega + i\alpha U)z^3) + z^4_Y]\exp(i\theta) = 0,$$

$$L^4(\exp(i\theta)z^4) \equiv [(-i\omega + i\alpha U)\varrho + \rho_Y z^3 + \rho(i\alpha z^1 + z^3_Y)]\exp(i\theta) = 0, \tag{4}$$

$$L^5(\exp(i\theta)z^5) \equiv [z^6 - z^5_Y]\exp(i\theta) = 0,$$

$$L^6(\exp(i\theta)z^6) \equiv [\rho((-i\omega + i\alpha U)z^5 + T_Y z^3) + (\gamma - 1)(i\alpha z^1 + z^3_Y) - \mu\gamma/(\sigma\,\mathrm{Re})z^6_Y]\exp(i\theta) = 0,$$

$$\varrho = \rho(z^4/P - z^5/T).$$

Equations (4) are the linearized Dan–Lin system [9] for two-dimensional disturbances. Here $\gamma = C_P/C_V$ is the ratio of specific heats, $\sigma = C_P\mu/K$ is the Prandtl number, and K is the thermal conductivity. The Mach and Prandtl numbers are based on the flow parameters outside the boundary layer, and the nonlinear terms in Eq. (3) are written in the following form:

$$F^1 = 0, \quad F^2 = \rho(u'u'_X + v'u'_Y) + \varrho(u'_t + Uu'_X + U_Y v'), \quad F^3 = \rho(u'v'_X + v'v'_Y) + \varrho(v'_t + Uv'_X),$$

$$F^4 = \varrho(u'_X + v'_Y) + u'\varrho_X + v'\varrho_Y, \quad F^5 = 0,$$

$$F^6 = \varrho(\Theta_t + U\Theta_X + T_Y v') + \rho(u'\Theta_X + v'\Theta_Y) + 2\gamma(\gamma - 1)\mathrm{M}^2 p'(u'_X + v'_Y).$$

The nonlinear effects are determined by summands with a quadratic amplitude in nonlinear terms.

The boundary conditions for disturbances are

$$z^1 = z^3 = z^5 = 0, \qquad Y = 0, \qquad Y = \infty. \tag{5}$$

Systems (3) and (4) are solved by the method of orthogonalization [9].

In the first order in terms of ε, the homogeneous system (4) is the basis for finding the eigenvalues of α for given values of the frequency ω and Reynolds number Re and also for constructing amplitude functions of linear waves of the form (2) with an undetermined amplitude parameter A with normalization $|z^3|_{\max} = 1$. In the weakly nonlinear theory, these parameters of linear waves are considered to be unknown (sought), and the nonlinearity affects the amplitude A only.

In the second order in terms of ε, the system of inhomogeneous differential equations (3) is used to construct secondary harmonics. We study the characteristics of secondary waves.

Self-action of initial waves of the form (2) leads to the appearance of summands of the form $A_jA_{-j}u_ju_{-j}\exp(i\theta_j - i\theta_j)$ and $A_jA_ju_ju_j\exp(i\theta_j + i\theta_j)$ $(j = 1, 2)$ in nonlinear terms $F^k(u'_j, u'_j)$. The first of them take into account induction of zero secondary harmonics with zero phases $\theta_{j,-j} = 0$. We denote their amplitude functions as $z^k_{j,-j}$. Being definitely steady, zero secondary harmonics contribute to distortion of the mean flow characteristics U and T. The force field created by the second summands in F^k generates overtones with doubled phases $\theta_{j,j} = 2\theta_j$. We denote the amplitude functions of overtones as $z^k_{j,j}$.

Combinatorial interaction of u'_1 and u'_2 leads to the appearance of both the secondary summary wave with the amplitude function $z^k_{1,2}$ and total phase $\theta_{1,2} = \theta_1 + \theta_2$ and the secondary difference wave with the amplitude function $z^k_{1,-2}$ and phase $\theta_{1,-2} = \theta_1 - \theta_2$. Thus, we have to consider six secondary harmonics for two waves.

The general solution for secondary waves is found using the generic scheme $z^k = Cz^k_{\mathrm{h}} + z^k_{\mathrm{inh}}$, where z^k_{h} are the solutions of the homogeneous system (4) and z^k_{inh} are the particular solutions of the inhomogeneous system (3). We can naturally assume that the eigenfunctions of secondary waves also satisfy the boundary conditions (5). The

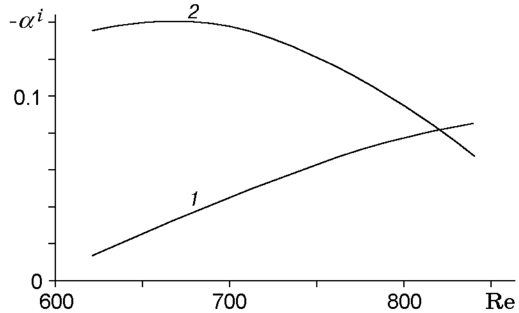

Fig. 1. Growth rates of linear waves: curves 1 and 2 refer to the frequencies f_1 and f_2, respectively.

system of amplitude equations for this type of nonlinear relations can be obtained using the standard averaging procedure and solvability conditions [7, 8]:

$$\frac{dA_1}{d\xi} = \{-\alpha_1^i + [E_{1,1,-1}|A_1|^2 + E_{1,2,-2}|A_2|^2]Z_1{}^{-1}\}A_1,$$
$$\frac{dA_2}{d\xi} = \{-\alpha_2^i + [E_{2,2,-2}|A_2|^2 + E_{2,1,-1}|A_1|^2]Z_2{}^{-1}\}A_2. \qquad (6)$$

Here $Z_j = \int\limits_Y \Big\{ \sum^k (z_j^k)^+ \dfrac{\partial L^k(z_j^k)}{\partial \alpha_j} \Big\}\, dY$; the superscript plus indicates the solutions of the system conjugate to (4). The coefficients E characterize the nonlinear relation of primary waves and secondary harmonics:

$$E_{1,1,-1} = \int\limits_Y \Big\{ \sum^k (z_1^k)^+ \frac{F^k(z_1^k, z_{1,-1}^k) + F^k(z_{-1}^k, z_{1,1}^k)}{\partial L^k(z_1^k)/\partial\omega_1} \Big\}\, dY,$$

$$E_{1,2,-2} = \int\limits_Y \Big\{ \sum^k (z_1^k)^+ \frac{F^k(z_1^k, z_{2,-2}^k) + F^k(z_{-2}^k, z_{1,2}^k) + F^k(z_2^k, z_{1,-2}^k)}{\partial L^k(z_1^k)/\partial\omega_1} \Big\}\, dY.$$

The term $E_{1,1,-1}$ takes into account the self-action of the wave: effect of distortion of the mean field $z_{1,-1}^k$ and overtone $z_{1,1}^k$ on the amplitude of the first wave. The second term $E_{1,2,-2}$ takes into account the additional effect of distortion of the mean field generated by the second wave and also the influence of the secondary summary and difference waves on the amplitude A_1 of the first wave. The amplitude equation for the second wave has the same structure.

We write the complex amplitudes A_j in a trigonometric form

$$A_j = a_j \exp(i\psi_j) \qquad (a = |A|, \quad \psi = \arg A)$$

and solve Eqs. (6) with respect to a and ψ. The initial values of $a_j(\xi_0)$ for Eqs. (6) were set via the initial wave intensities I_j. In this case, we have $\xi_0 = X_0$, where X_0 is determined by the initial value of Re_0 (see below). The relation between the disturbance amplitude and intensity is expressed in terms of the maximum (along the transverse coordinate Y) calculated value of mass-velocity fluctuations $m' = \rho u + \varrho U$ of the subharmonic ($j = 1$)

$$I_j(\xi_0) = a_j(\xi_0) m'_{1\max} \exp(-\alpha_j^i \xi_0)$$

and the initial phases can be chosen arbitrarily [$\psi_j(\xi_0) = 0$].

Results and Discussion. In the experiments [1], the position of the source of controlled disturbances corresponded to the value Re = 497; the measurements were performed within the range $\mathrm{Re}_0 \leqslant \mathrm{Re} \leqslant 846$, $\mathrm{Re}_0 = 624$. The stagnation temperature in the experiments was constant and reached 310 K; $\gamma = 1.4$ and $\sigma = 0.72$. The calculations were performed for the same parameters.

Figure 1 shows the growth rates $-\alpha^i$ of linear waves with the subharmonic frequency f_1 and fundamental frequency f_2 (curves 1 and 2, respectively). The initial position of the subharmonic for $\mathrm{Re} = \mathrm{Re}_0$ is the region near the lower branch of the neutral curve; the linear growth rate increases with increasing Re and does not reach

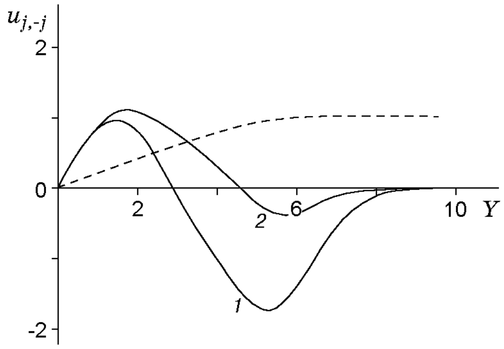

Fig. 2

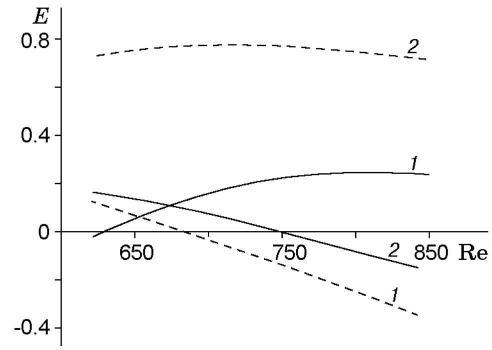

Fig. 3

Fig. 2. Amplitude functions of zero secondary harmonics for Re = Re_0: curves 1 and 2 refer to $u_{1,-1}$ and $u_{2,-2}$; the dashed curve is the mean undisturbed profile of the streamwise velocity U.

Fig. 3. Nonlinear coefficient versus Re for waves with the frequencies f_1 (curve 1) and f_2 (2): the solid and dashed curves refer to the regimes of self-action and combinatorial interaction, respectively.

a maximum in the last measurement section. For Re = Re_0, the main wave is located near the maximum of the linear growth rate, which decreases downstream.

Among the secondary harmonics, of greatest interest are the zero harmonics ($u_{1,-1}$ and $u_{2,-2}$), which determine the deformation of diagrams of the averaged streamwise velocity found experimentally. Figure 2 shows their transverse distributions (curves 1 and 2) for Re = Re_0 and also the mean undisturbed profile U (dashed curve). The total deformation of U is written in the form $\Delta U = |a_1|^2 u_{1,-1} + |a_2|^2 u_{2,-2}$ and is determined by the amplitudes of initial waves. Thus, the allowance for nonlinearity is responsible for the greater filling of the profile in the near-wall region and the appearance of a velocity defect near the external boundary, which increases the boundary-layer thickness. This result is in good agreement with experimental data.

The influence of nonlinear processes on the amplitude of primary waves can be analyzed by considering nonlinear coefficients. We unite the coefficients that take into account self-action and combinatorial interaction [see Eq. (7)]. It follows from Eqs. (6) that the amplitudes a_j are determined by the real values of these coefficients (Fig. 3). The positive values of nonlinear terms lead to an additional [as compared to the linear value $a(\xi_0)\exp(-\alpha^i X)$] increase in amplitudes, whereas the negative values lead to an amplitude decrease. It follows from Fig. 3 that the self-action of the subharmonic (solid curve 1) increases the amplitude a_1, and the self-action of the wave with the frequency f_2 (dashed curve 1) decreases the amplitude. The effect of the main wave on the subharmonic $E_{1,2,-2}$ is manifested in destabilization of the subharmonic amplitude for low values of Re; this influence decreases significantly in the middle of the range examined (solid curve 2). For the wave with the fundamental frequency f_2, the presence of a subharmonic in the spectrum always leads to an increase in a_2 (dashed curve 2).

Thus, the influence of nonlinearity in the process considered is ambiguous; it can become stronger or weaker and, as a whole, depends on the values of the amplitudes under consideration.

We analyze the behavior of the amplitudes within the framework of the model considered (Fig. 4). The initial values of a_1 and a_2 in calculations corresponded to experimental summary intensities of the initial wave packets; for $\xi = \xi_0$, we had $I_1/I_2 = 3$ and $a_1/a_2 = 2$.

Nonlinearity always leads to a more intense increase in the subharmonic amplitude. The ratio of final and initial amplitudes is approximately equal to 3 in the linear model and to 6 in the nonlinear approximation. Thus, the amplitude can increase almost twofold due to nonlinearity. The main contribution to the increase in amplitude is made by self-action; the presence of the wave with the fundamental frequency leads to a decrease in amplitude because part of energy is transferred to this wave.

For the main wave, the linear amplitude a_2 decreases because of a significant decrease in linear growth rates within the range of Re under study; the allowance for nonlinearity in the regime of self-action has practically no effect on the amplitude, and the presence of the subharmonic leads to an insignificant increase in amplitude. The weak effect of nonlinearity is determined by the small value of the initial amplitude of the wave.

The process of disturbance propagation in the boundary layer depends on the values of initial amplitudes and their ratio a_1/a_2.

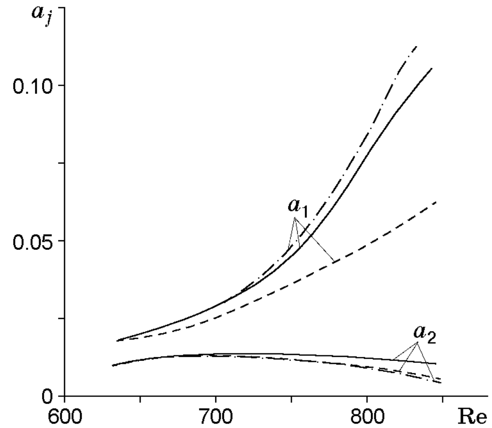

Fig. 4

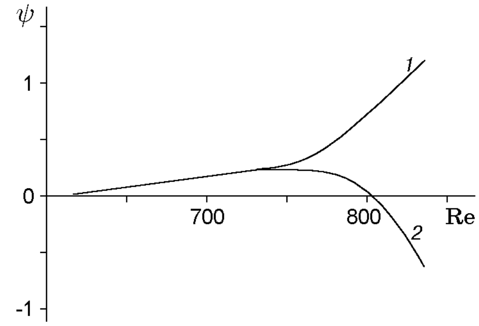

Fig. 5

Fig. 4. Dynamics of the absolute values of the wave amplitudes a_1 and a_2: the solid curves refer to the model of combinatorial interaction, the dot-and-dashed curves show the regime of self-action, and the dashed curves refer to the linear model.

Fig. 5. Phase incursion of the complex amplitudes $A = a\exp(i\psi)$ in the nonlinear process for waves with the frequencies f_1 (curve 1) and f_2 (2).

The dynamics of the absolute values of the amplitude was considered below. The nonlinear process includes also the phase incursion ψ_j, which can change the wavenumbers α_j^r and, hence, the phase velocities $c_j = \omega_j/\alpha_j^r$. It was noted in [1] that the values of phase velocities $c = 0.70$–0.72, on the average, were 30–40% higher than the phase velocities of own waves in a supersonic boundary layer $c = 0.52$–0.55. Figure 5 shows the phase incursion obtained in calculations. The dependence $\psi_j(\mathrm{Re})$ demonstrates the effectiveness of wave interaction. It follows from Fig. 5 that the role of nonlinearity increases in the middle of the examined range of Reynolds numbers. In this range, the values of $\Delta\alpha_j^r = d\psi_j/d\,\mathrm{Re}$ were found, and the phase velocities of the waves at the nonlinear stage were determined. The phase velocities with nonlinear corrections ($c = 0.69$–0.75) are close to experimental values.

Thus, the combinatorial model considered yields a qualitatively correct description of the special features of the dynamics of high-intensity controlled disturbances: significant amplification of the signal as compared to the linear case, distortion of mean-velocity profiles near the external boundary, which leads to an increase in the boundary-layer thickness, and increase in phase velocities of the waves.

The results presented indicate that the nonlinear process considered may occur in the boundary layer excited by a controlled high-intensity signal. The interaction of real wave packets rather than solitary waves should be further examined, and steady vortex modes, which may affect the nonlinear process, should be analyzed.

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